REDUCED ORDER MODELLING OF CONTINUOUS TIME SISO SYSTEMS

By

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ABSTRACT

Many systems that arise in practice are complex in nature and of a higher-order. The mathematical procedures of modeling such systems lead to a comprehensive description of the process in the form of complex, higher-order transfer functions or state-space models. This complexity often makes it difficult to obtain a good understanding of the behavior of the system. Therefore, higher-order models are difficult to use for simulation, analysis or controller synthesis, and it is not only desirable, but often necessary to obtain satisfactory reduced-order representations of such higher-order models. The main objective of model order reduction is to obtain a reduced-order approximate of a complex higher-order system that retains and reflects the important characteristics of the original system as closely as possible.

Keywords: Reduced Order Model, CRA (Characteristic Ratio Assignment), AGTM (Approximate Generatized Time Management), SISO (Single Input Signal Output) Systems.

INTRODUCTION

Many modern mathematical models of real-life processes pose challenges when used in numerical simulations, due to complexity and large size (dimension). Model Order Reduction (MOR) aims to lower the computational complexity of such problems, for example, in simulations of large-scale dynamical systems and control systems [1], [3]-[5]. By a reduction of the model's associated state space dimension or degrees of freedom, an approximation to the original model is computed. This Reduced Order Model (ROM) can then be evaluated with lower accuracy, but in significantly less time. This paper presents some results and approaches to directly address the transient response control problem and also steady state response control problem [10]-[14]. The main ideas are based on certain relations between characteristic polynomial coefficients and time domain responses [1]-[2]. New techniques were also implemented for reducing the model of complex systems.

This research work is undertaken in a two-fold manner; firstly, to present new methods for obtaining the reduced order models for high order linear time-invariant dynamic systems, continuous domain, and secondly, to apply the model order reduction philosophy to the design of controllers for such systems [1], [3], [8]-[12]. The methods have developed, mainly to use the transfer function description and are applicable to Single-Input Single-Output (SISO) as well as Multi-Input Multi-Output (MIMO) systems [1], [3], [5]. Some new methods have been developed for Model Order Reduction that attempt to overcome some of the inherent drawbacks of the prevalent techniques [3], [5], [10].

1. Reduction of Model Techniques

1.1 Transient Response Control via Characteristic Ratio Assignment (CRA)

This is an approach to directly control the transient response of linear time-invariant control systems [1], [12]-[13]. The main ideas are based on certain relations between characteristic polynomial coefficients and time domain responses [1]-[2]. In this paper, the authors have begun by defining two important sets of parameters called generalized time constant and characteristic ratios, [1]-[4]. These parameters are written in terms of the coefficients of a polynomial [1]-[5]. The properties of these parameters with respect to time domain response, in particular, speed of response and overshoot, are then derived analytically [2]-[12]. These properties are later used to construct a desired transfer function and a controller design procedure for minimum phase plants to achieve a transient response, with independently specified overshoot and rise time [2], [9] - [10].

1.1.1 Speed of Step Response

If, x(t) is a signal, w(t) = x(βt) is a time scaled version of x(t) and is speeded up if β > 1, and slowed down, if 0 < β < 1. Let, y(t) denote the forced response, to a step r(t), of a system with transfer function G(s). The authors are interested in determining a system with transfer function, H(s) so that, its forced response to r(t) is y(βt) for a given β > 0. H(s) speeds up the step response of G(s) by a factor β, [1]-[3]. Let,

$$G(s)=\frac{N(s)}{D(s)}=\frac{n_m s^m+n_{m-1}s^{m-1}+\cdots+n_1 s+n_0}{d_n s^n+d_{n-1}s^{n-1}+\cdots+d_1 s+d_0}=\frac{K(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)} \quad (1)$$

$$H(s)=\frac{N_H(s)}{D_H(s)}=\frac{a_m s^m+a_{m-1}s^{m-1}+\cdots+a_1 s+a_0}{b_n s^n+b_{n-1}s^{n-1}+\cdots+b_1 s+b_0}=\frac{\hat{K}(s-\hat{z}_1)(s-\hat{z}_2)\cdots(s-\hat{z}_m)}{(s-\hat{p}_1)(s-\hat{p}_2)\cdots(s-\hat{p}_n)} \quad (2)$$

1.1.2 Theorem 1

Given G(s) as before, H(s) which speeds up the step response of G(s) by a factor β, is uniquely determined by one of the following equivalent conditions [1], [3]-[5]:

$$a_i = n_i/\beta^i, i=0,1,2,\ldots,m \text{ And } b_j = d_j/\beta^i, j=0,1,2,\ldots,n \quad (3)$$

$$\hat{z}_i = \beta z_i, i=0,1,2,\ldots,m \text{ And } \hat{p}_i = \beta p_i, i=0,1,2,\ldots,n, \hat{K}=\beta^{n-m}K \quad (4)$$

Proof:

Introducing the characteristic ratios and generalized time constant for a Hurwitz polynomial [1]-[3]:

$$p(s)=a_n s^n+a_{n-1}s^{n-1}+\cdots+a_1 s+a_0, \quad a_i>0 \quad (5)$$

Characteristic ratios is given by,

$$\alpha_1=\frac{a_1^2}{a_0 a_2} \quad \alpha_2=\frac{a_2^2}{a_1 a_3},\ldots,\alpha_{n-1}=\frac{a_{n-1}^2}{a_{n-2}a_n} \quad (6)$$

and the generalized time constant is given by,

$$\tau=\frac{a_1}{a_0} \quad (7)$$

If G(s) is stable and minimum phase, then n_i, d_i > 0 for all i's without loss of generality, and the characteristic ratios of N(s) and D(s) are defined accordingly. Let the characteristic ratios of N(s), D(s), N_H(s), and D_H(s) be α_i^N, α_i^D, $\alpha_i^{N_H}$ and $\alpha_i^{D_H}$ respectively. Similarly, let the generalized time constants of those polynomials be, τ^N, τ^D, τ^{N_H} and τ^{D_H}, respectively.

If G(s) is stable and minimum phase, H(s) is uniquely determined by,

$$\alpha_i^{N_H}=\alpha_i^N, i=0,1,2,\ldots,m-1 \text{ And } \alpha_j^{D_H}=\alpha_j^D, j=0,1,2,\ldots,n-1 \text{ and} \quad (8)$$
$$\tau^{N_H}=(1/\beta)\tau^N, \tau^{D_H}=(1/\beta)\tau^D, a_0/b_0=n_0/d_0$$

Remarks:

1) Equation (3) shows how the coefficients of H(s) can be obtained by scaling the corresponding coefficients of G(s).
2) Equation (4) shows that the poles and zeros of G(s) are moved out along rays from the origin by a factor of β, while the "dc" gain G(0) = H(0).
3) Equation (8) shows that the characteristic ratios of the numerators and denominators of H(s) remain unchanged, respectively, from those of G(s), while the generalized time constant of both numerator and denominator are reduced by a factor β.

1.1.3 Theorem 2

Let G(s) be an all-pole transfer function [1]-[5]:

$$G(s)=\frac{a_0}{p(s)}=\frac{a_0}{a_n s^n+a_{n-1}s^{n-1}+\cdots+a_1 s+a_0}, a_i>0 \quad (9)$$

and let α_i be the characteristic ratios of p(s). Then,

1) The frequency magnitude function |G(jω)| is monotonically decreasing and
2) p(s) is Hurwitz; if the following two conditions hold:

$$\alpha_1>2; \quad \alpha_k=\frac{\sin\left(\frac{k\pi}{n}\right)+\sin\left(\frac{\pi}{n}\right)}{2\sin\left(\frac{k\pi}{n}\right)}\alpha_1, \text{ for } k=2,3,\ldots,n-1 \quad (10)$$

Proof:

The detailed proofs can be found in [1]. To prove Theorem 2, the following lemma are necessary in describing the properties of Γ_k.

Lemma 1: For the definition of Γ_K in equation (7), we have the following

i) $\Gamma_1 = \Gamma_{n-1}, \Gamma_2 = \Gamma_{n-2}, \ldots\ldots$

$$\Gamma_{m-1} = \begin{cases}\Gamma_m, & if\ n=2m-1\\ \Gamma_{m+1}, & if\ n=2m\end{cases} \quad (11)$$

ii) Consider two polynomials of degrees n_1 and n_2 (n_1 > n_2) satisfying the equations (3) and (4) of Theorem 2 and let $\Gamma_k(n_1)$ and $\Gamma_k(n_2)$ be the corresponding Γ_k at n=n_1 and n=n_2, respectively. Then,

$$\Gamma_k(n_1) \le \Gamma_k(n_2), \text{ for all k} \quad (12)$$

and equality holds at k=1.

iii)

$$\min_n \Gamma k = \lim_{n\to\infty} \Gamma k = \frac{k+1}{2k} \quad (13)$$

Proof of Result-1:

Consider the square of frequency magnitude of a stable all-pole transfer function,

$$G(j\omega)|^2 = \frac{1}{\frac{1}{a^2_0} \cdot p(j\omega)p(-j\omega)} = \frac{1}{q(w)} \qquad (14)$$

Using the relationships given in equations (2) and (3), q(w) is given in terms of the characteristic ratios α_i and generalized time constant τ of p(s).

$$\Delta lk = \Pi_{i=k}^{l} l\propto = \alpha_k \ \alpha_{k+1} \ldots\ldots \alpha_1, \text{ for } k\leq l \qquad (15)$$

Then,

$$q(w)=1 + \frac{\Delta_1^1-2}{\Delta_1^1} \cdot \tau^2\omega^2 + \frac{\Delta_2^2\Delta_1^3-2\Delta_1^3+2}{\Delta_2^2\ \Delta_1^3(\Delta_1^1)^2} \cdot \tau^4\ \omega4 + \ldots\ldots + \frac{1}{(\Delta_1^1\Delta_1^3\Delta_1^3 \ldots \Delta_1^{n-1})^2} \cdot \tau^{2n}\omega^{2n}. \qquad (16)$$

Result-1 is proved by showing that, the even degree function in equation (16) is a monotonically increasing function. The function q(w) is monotonically increasing if all the coefficients are positive. Thus, since all 's Δ_i^j are positive, q(w) is a monotonically increasing function if,

$$\eta_1 = \Delta_1^1 - 2 > 0$$
$$\eta_2 = \Delta_2^2\Delta_1^3 - 2\Delta_1^3 + 2 > 0 \qquad (17)$$
$$\eta_{n-1} = \Delta_{n-1}^{n-1} - 2 > 0.$$

The proof is completed by showing that, equation (17) holds under the conditions of equations (3) and (4) in Theorem 2. It is not difficult to show by using lemma 1 that for k=1,2,3,…………, n-1,

$$\eta_k = 0, \text{ for } \alpha_1 = 2 \qquad (18)$$
$$\frac{d\eta_k}{d\alpha_1} > 0, \text{ for all } \alpha_1 > 2.$$

This concludes the proof of the result (1) in Theorem 2.

Proof of Result-2:

Note that p(s) is Hurwitz for $\alpha_1 = 2$. Consequently, the n^{th} order polynomial image p(jω) $|\alpha_1=2$ obeys the monotonic phase increase the property and turns $n\pi/2$ over $\omega\in(0,\infty)$ by the Mikhailov criterion [3]. Thus, it is enough to show that the phase of p(jω) is monotonically increasing over $\omega\in(0,\infty)$, for all $\alpha_1 > 2$. From equations (11), (12),and (15), with $\tilde{\omega}= t\omega$,

$$p(j\omega) = h(-\tilde{\omega}^2) + j\tilde{\omega}\, g(-\tilde{\omega}^2) \qquad (19)$$

where,

$$h(-\hat{\omega}^2) = a_0 \left\{1 - \frac{1}{\Delta_1^1}\tilde{\omega}^2 + \frac{1}{\Delta_1^3\Delta_1^2\Delta_1^1}\tilde{\omega}^4 - \frac{1}{\pi_{j=1}^{5}\Delta_i^j}\tilde{\omega}^6 + \cdots\ldots\right\} \qquad (20)$$

$$G(-\hat{\omega}^2) = a_0 \left\{1 - \frac{1}{\Delta_1^2\Delta_1^1}\tilde{\omega}^2 + \frac{1}{\pi_{j=1}^{4}\Delta_1^j}\tilde{\omega}^4 - \frac{1}{\pi_{j=1}^{6}\Delta_1^j}\tilde{\omega}^6 + \cdots\ldots\right\} \qquad (21)$$

Define

$$\Phi(\tilde{\omega}) = \tan \frac{\tilde{\omega}\, g(-\tilde{\omega}^2)}{h(-\tilde{\omega}^2)} = \Phi(\omega). \qquad (22)$$

The monotonic phase increase property [3] of p(s) $\frac{d\Phi(\omega)}{d(\omega)} > 0$, for all $\omega \geq 0$ is equivalent to,

$$L_n(\tilde{\omega}) = g(-\tilde{\omega}^2)h(-\tilde{\omega}^2) + \tilde{\omega}\left[\frac{dg(-\tilde{\omega}^2)}{dw}h(-\tilde{\omega}^2) - g(-\tilde{\omega}^2)\frac{dh(-\tilde{\omega}^2)}{dw}\right] > 0 \text{ for all } \omega \geq 0 \qquad (23)$$

Rewriting equation (23), we have for n odd (i.e., n=2m+1)

$$L_n(\tilde{\omega}) = 1 + \frac{\mu_1}{\Delta_1^2\Delta_1^1}\tilde{\omega}^2 + \frac{\mu_2}{\pi_{j=1}^4\Delta_1^j}\tilde{\omega}^4 + \ldots\ldots + \qquad (24)$$
$$\frac{\mu_k}{\pi_{j=1}^{2k}\Delta_1^j}\tilde{\omega}^{2k} + \cdots\ldots + \frac{1}{(\pi_{j=1}^{2m}\Delta_1^j)(\pi_{i=1}^{2m-1}\Delta_1^i)}\tilde{\omega}^{(n-1)^2} \qquad (k\leq m)$$

where for k=1, 2, 3,……,n-2. equation (26), shown below, holds. From the expression for the denominator of each term in equation (24), the phase monotonicity equation (23) is satisfied if all μ l 's in equation (25) are positive. From property, i) in Lemma 1,

$$\mu_1 = \mu_{n-2},\ \mu_2 = \mu_{n-3}\ldots\ldots$$

$$\mu_{m-1} = \begin{cases}\mu_m, & \text{if } n = 2m \\ \mu_{m+1}, & \text{if } n = 2m+1\end{cases} \qquad (25)$$

$$\mu_k = \begin{cases}\pi_{j=0}^{k-1}\Delta_{k-j}^{k+j+1} - 3\pi_{j=1}^{k-1}\Delta_{k-j}^{k+j+1} + \cdots ..+(-1)^k(2k+1), & \text{for } k\leq m \\ \pi_{j=0}^{2m-k-1}\Delta_{k-j}^{k+j+1} - 3\pi_{j=1}^{2m-k-1}\Delta_{k-j}^{k+j+1} + \cdots ..+(-1)^k\{2(2m-k)-1\} & \text{for } k\geq m+1\end{cases} \qquad (26)$$

Therefore, it is enough to show that $\mu_l > 0$ for l = 1,2,…., m or m-1. From property ii) in lemma 1, we also observe that the following is true.

$$\Delta_k^l(n_1) < \Delta_k^l(n_2), \text{ for } l > k \qquad (27)$$

where,

$$\Delta_k^l(n) = \pi_{i=k}^{l}\alpha_i = \Gamma l(n)\Gamma_{l-1}(n)\ldots\Gamma_k(n)\,\alpha_1^{l-k+1} \qquad (28)$$

Under the condition in equation (26) ,it is clear that if, $\mu_l(n_l)$ for all l are positive, $\mu_i(n)$ for all i are also positive for any such a polynomial of degree $n < n_1$. Thus, the proof of the result (2) is accomplished by showing that, $\mu_i(n)$ for all i are positive when, $n \rightarrow \infty$.

It can be shown by using equation (26) and (28) that for k = 1,2,3,…..,n-2,

$$\mu_k = 0, \text{ if } \alpha_1 = 2$$
$$\frac{d\mu_k}{d\alpha_1} > 0, \text{ for all } \alpha_1 > 2. \qquad (29)$$

Thus the proof is completed.

The authors establish this theorem by showing that, all $\eta_i \geq 0$ when all $a_i > 2$. Recall equation (17) and write,

$$\eta_1 = (\alpha_1 - 2)$$

$$\eta_2 = (\alpha_2 - 2)\Delta_1^3 + 2$$

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$$\eta_{n-1} = (\alpha_{n-1} - 2) \qquad (30)$$

Observe that,

$$\Delta_k^l = \alpha_k \alpha_{k-1} \;\ldots\ldots\; \alpha_1 > 1, \text{ for all } > \alpha_i 2; \text{ any } k \leq l. \qquad (31)$$

Thus, for $\alpha_1 > 2$, $l = 1,2,3,\ldots,n-1$, every η_k is positive.

The coefficients of p(s) is calculated from characteristic ratios and time constant, as follows:

$$a_1 = \tau a_0 \qquad (32)$$

$$a_i = \frac{\tau^i a_0}{\alpha_{i-1}\alpha_{i-2}^2\,\alpha_{i-3}^3 \ldots \alpha_1^{i-1}}, \quad \text{for } i = 2,3,\ldots,n \qquad (33)$$

Let us consider three all pole transfer functions, $T_k(s) = a_0/P_k(s)$ of different orders, but having the same α_1 and α_2, which are chosen to obtain sufficiently large $\alpha_1 \times \alpha_2$. All parameters and coefficients are shown in Table 1, and the step responses of these transfer functions are shown in Figure 1. Therefore the theorem is proved.

Even though the three transfer function models are quite different, except having similar values of α_1, α_2 and τ; surprisingly, they have almost the same step responses. This result is caused by the dominance of the characteristic ratios α_1 and α_2. This idea helps in reducing the order of the denominator of a transfer function that has dominant characteristic ratios α_1 and α_2.

	n	$\alpha = [\alpha_1\ \alpha_2 \ldots \alpha_{n-1}]$	τ	Coefficients of $p_k(s)$ $[a_n \ldots a_1\ a_0]$
$T_1(s)$	3	[0.6 6]	10	[462.9 166.67 10 1]
$T_2(s)$	5	[0.6 6 1.5 1.5]	10	[1058 857.3 462.9 166.67 10 1]
$T_3(s)$	10	[0.6 6 1.5 1.5...1.5 1.5]	10	[6.932 42.64 174.8 478 871.1 1058 857.3 462.9 166.67 10 1]

Table 1. Parameters of Three Transfer Functions

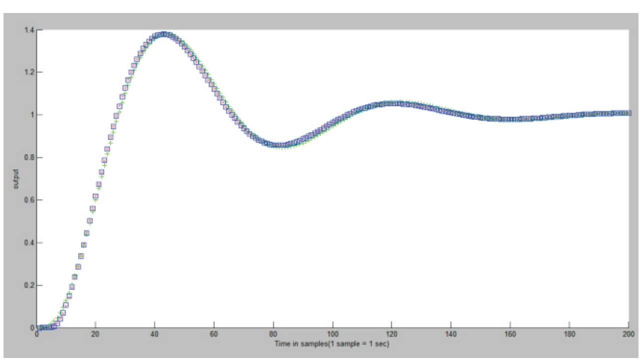

Figure 1. Step Response Comparison (Table 1)

Remarks

1) The authors experience also shows that increasing α_1 reduces overshoot.
2) Reducing order of the denominator of transfer function which is having dominant characteristic ratios α_1 and α_2.

This idea of characteristic ratio assignment can be used to reduce the order of the denominator of a higher-order of all-pole transfer function. Now, the focus will be to find methods to reduce the order of numerator of a general transfer function so that, the step response of the reduced transfer function matches the original system as closely as possible.

2. ROD Methods

To reduce the order of numerator, the authors have considered four methods, of which first three methods use the CRA technique to find the Reduced Order Denominator (ROD). The methods are [5], [6], [8]-[10]:

1. Time moment matching.
2. Time moment and Markov parameter matching.
3. Approximate Generalized Time Moment (AGTM) matching.
4. AGTM matching for obtaining the complete Reduced Order Model (ROM).

Consider a SISO system described by a transfer function of order 'n'.

$$G_n(s) = \frac{N(s)}{D(s)} = \frac{a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \cdots + a_1 s + a_0}{s^n + b_{n-1}s^{n-1} + \cdots + b_1 s + b_0} \qquad (34)$$

The problem is to determine its stable reduced-order (r^{th}-order) approximate:

$$G_r(s) = \frac{N_r(s)}{D_r(s)} = \frac{\hat{a}_{r-1}s^{r-1} + \hat{a}_{r-2}s^{r-2} + \cdots + \hat{a}_1 s + \hat{a}_0}{s^r + \hat{b}_{r-1}s^{r-1} + \cdots + \hat{b}_1 s + \hat{b}_0} \qquad (35)$$

$G_n(s)$ is a given transfer function and $D_r(s)$ is found by obtaining reduced order denominator D(s) of the original transfer function G(s) through the CRA technique. The order 'r' of $D_r(s)$ is the minimum 'r' value up to which the step response of transfer function $(1/D_r(s))$ tracks the step response of transfer function (1/D(s)) as closely as possible.

2.1 Complete Time Moment Matching [5]-[8]

Expanding $G_n(s)$ around s = 0:

$$G_n(s)=\frac{N(s)}{D(s)}=\frac{a_{n-1}s^{n-1}+a_{n-2}s^{n-2}+\cdots+a_1s+a_0}{s^n+b_{n-1}s^{n-1}+\cdots+b_1s+b_0}=t_0+t_1s+\cdots+t_{n-1}s^n+\cdots \quad (36)$$

To find the reduced numerator, equate $G_r(s)$ to the above expansion:

$$G_r(s)=\frac{\hat{a}_{r-1}s^{r-1}+\hat{a}_{r-2}s^{r-2}+\cdots+\hat{a}_1s+\hat{a}_0}{s^r+\hat{b}_{r-1}s^{r-1}+\cdots+\hat{b}_1s+\hat{b}_0}=t_0+t_1s+\cdots+t_{r-1}s^{r-1} \quad (37)$$

Multiplying the above equation with $D_r(s)$ (known from CRA) on both sides, and equating the coefficients of equal power of 's'; after simplification, the reduced numerator coefficients are obtained by solving the following equations:

$$\begin{aligned}\hat{a}_0 &= \hat{b}_0t_0\\ \hat{a}_1 &= \hat{b}_0t_1+b_1t_0\\ &\vdots\\ \hat{a}_{r-1} &= \hat{b}_0t_{r-1}+\hat{b}_1t_{r-2}+\cdots+\hat{b}_{r-1}t_0\end{aligned} \quad (38)$$

The numerator coefficients of the reduced model obtained by this method ensures good low frequency (large time) matching i.e., steady state matching of the original system and reduced order model because the expansion of $G_n(s)$ has been done around s=0.

2.2 Time Moment and Markov Parameter Matching [5], [6]-[8]

Expanding $G_n(s)$ around s = ∞, one obtains:

$$G_n(s)=\frac{N(s)}{D(s)}=\frac{a_{n-1}s^{n-1}+a_{n-2}s^{n-2}+\cdots+a_1s+a_0}{s^n+b_{n-1}s^{n-1}+\cdots+b_1s+b_0}=m_1s^{-1}+m_2s^{-2}+\cdots+m_ns^{-n}+\cdots \quad (39)$$

Equating $G_r(s)$ to the above expansion, the reduced numerator polynomial can be obtained. Thus,

$$G_r(s)=\frac{\hat{a}_{r-1}s^{r-1}+\hat{a}_{r-2}s^{r-2}+\cdots+\hat{a}_1s+\hat{a}_0}{s^r+\hat{b}_{r-1}s^{r-1}+\cdots+\hat{b}_1s+\hat{b}_0}=m_1s^{-1}+m_2s^{-2}+\cdots+m_rs^{-r} \quad (40)$$

Multiplying the above equation with $D_r(s)$ (known from CRA) on both sides, and equating the coefficients of equal powers of 's'; after simplification, the reduced numerator coefficients are obtained by solving the following equations:

$$\begin{aligned}\hat{a}_{r-1} &= m_1\\ \hat{a}_{r-2} &= m_2+m_1\hat{b}_{r-1}\\ \hat{a}_{r-3} &= m_3+m_2\hat{b}_{r-1}+m_1\hat{b}_{r-2}\\ &\vdots\\ \hat{a}_0 &= m_r+m_{r-1}\hat{b}_{r-1}+\cdots+m_1b_1\end{aligned} \quad (41)$$

The numerator coefficients of the reduced model obtained by this method ensures good high frequency (transient state) matching of the original and reduced transfer functions because expansion of $G_n(s)$ is taken around s=∞.

Now, we know the reduced numerator from both complete time moment matching and complete Markov parameter matching. In order to obtain a good overall matching of time responses of $G_r(s)$ to $G_n(s)$ in both transient and steady states, one should match α time moments and β Markov parameters, so that α + β = r, the number of unknown parameters in the numerator polynomial. The best values of α and β to be chosen, depends on the system to be reduced, and cannot be determined a-priori. This is an open problem for further research.

2.3 Matching AGTMs for obtaining Reduced Numerator [5]-[12]

In this method, we will equate the $G_r(s)$ and $G_n(s)$ at specific real values of 's'.

$$\Rightarrow G_r(s)\big|_{s=\delta_i}=G(s)\big|_{s=\delta_i}=G(\delta_i) \quad (42)$$

$$\Rightarrow N_r(s)\big|_{s=\delta_i}=G(\delta_i)*D_r(\delta_i) \quad (43)$$

where, δ=0.01 and δ_i=δ*i, for i =1,2,...,r for steady state matching; like time moment matching, and δ_i=(1/δ)*i, for i =1,2,...,r for transient state matching like Markov parameter matching.

$$\Rightarrow \left[\hat{a}_{r-1}s^{r-1}+\hat{a}_{r-2}s^{r-2}+\cdots+\hat{a}_1s+\hat{a}_0\right]_{s=\delta_i}=G(\delta_i)*D_r(\delta_i) \quad (44)$$

$$\Rightarrow\begin{bmatrix}1 & \delta_1 & \delta_1^2 & \cdot & \cdot & \delta_1^{r-1}\\ 1 & \delta_2 & \delta_2^2 & \cdot & \cdot & \delta_2^{r-1}\\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\ 1 & \delta_r & \delta_r^2 & \cdot & \cdot & \delta_r^{r-1}\end{bmatrix}*\begin{bmatrix}\hat{a}_0\\ \hat{a}_1\\ \cdot\\ \cdot\\ \hat{a}_{r-1}\end{bmatrix}=\begin{bmatrix}G(\delta_1)*D_r(\delta_1)\\ G(\delta_2)*D_r(\delta_2)\\ \cdot\\ \cdot\\ G(\delta_r)*D_r(\delta_r)\end{bmatrix} \quad (45)$$

This is in the form of A.x=B and the reduced numerator coefficients can be easily obtained by solving the set of appropriate linear algebraic equations. One should choose an appropriate combination of δ_i=δ*i and/or δ_i=(1/δ)*i values so that, total number of δ_i values = 'r'; thus ensuring a good overall time response approximation.

2. 4 AGTM matching for obtaining the complete ROM [12]-[14]

It is similar to the above method, but here both the numerator and denominator of reduced transfer function are taken as unknowns. So here we should choose total 2r number of δ_i values to find 'r' number of reduced denominator coefficients and 'r' number of reduced numerator coefficients. Proceeding as above and bringing the known terms to one side, and the unknown terms to other side, and on rearranging, the equation in matrix form will be obtained as below:

$$\Rightarrow \left[\hat{a}_{r-1}s^{r-1} + \hat{a}_{r-2}s^{r-2} + \cdots + \hat{a}_1 s + \hat{a}_0\right]_{s=\delta_i} = G(\delta_i) * \left[s^r + \hat{b}_{r-1}s^{r-1} + \cdots + \hat{b}_1 s + \hat{b}_0\right]_{s=\delta_i} \quad (46)$$

$$\Rightarrow \begin{bmatrix} 1 & \delta_1 & \delta_1^2 & \cdot & \cdot & \delta_1^{r-1} & -G(\delta_1) & -G(\delta_1)*\delta_1 & \cdot & \cdot & -G(\delta_1)*(\delta_1)^{r-1} \\ 1 & \delta_2 & \delta_2^2 & \cdot & \cdot & \delta_2^{r-1} & -G(\delta_2) & -G(\delta_2)*\delta_2 & \cdot & \cdot & -G(\delta_2)*(\delta_2)^{r-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \delta_r & \delta_r^2 & \cdot & \cdot & \delta_r^{r-1} & -G(\delta_{2r}) & -G(\delta_{2r})*\delta_{2r} & \cdot & \cdot & -G(\delta_{2r})*(\delta_{2r})^{r-1} \end{bmatrix} * \begin{bmatrix} \hat{a}_0 \\ \cdot \\ \hat{a}_{r-1} \\ \hat{b}_0 \\ \cdot \\ \hat{b}_{r-1} \end{bmatrix} = \begin{bmatrix} G(\delta_1)*(\delta_1)^r \\ G(\delta_2)*(\delta_2)^r \\ \cdot \\ \cdot \\ \cdot \\ G(\delta_{2r})*(\delta_{2r})^r \end{bmatrix} \quad (47)$$

The matrix solution of a set of linear algebraic equations are applied to the above matrix equation and the numerator and denominator coefficients of the ROM are obtained.

In the following, the authors have consider several examples from the literature for order reduction by using all the above four methods. The results will be compared to find the best reduction method. For comparison, a performance index is chosen as the sum of square of error in step responses of G(s) and $G_r(s)$ at the chosen sampled points. Let Y(t) and $Y_r(t)$ are the step responses of G(s) and $G_r(s)$. Then the performance index is taken as:

$$J = \sum_{t=1}^{m} (Y(t) - Y_r(t))^2 \quad (48)$$

where, m is the number of sampled points.

3. Simulation Results of Continuous Time SISO Systems

3.1 Example 1

The given high order transfer function is taken from [7] where,

$$G(S) = \frac{s\text{^}5 + 1014\,s\text{^}4 + 14069\,s\text{^}3 + 69140\,s\text{^}2 + 140100\,s + 100000}{s\text{^}6 + 222s\text{^}5 + 14541s\text{^}4 + 248420\,s\text{^}3 + 1.454e006\,s\text{^}2 + 2.22e006\,s + 1e006}$$

Reduced order transfer functions are obtained as:

1. Complete time moment matching gives:

$$G_r(s) = \frac{0.06914\,s\text{^}2 + 0.1401\,s + 0.1}{0.2484\,s\text{^}3 + 1.454\,s\text{^}2 + 2.22\,s + 1}, \; J = 0.21414*10\text{^}(-2) \quad (49)$$

2. Two Time moment and One Markov parameter matching gives:

$$G_r(s) = \frac{0.2484\,s\text{^}2 + 0.1401s + 0.1}{0.2484\,s\text{^}3 + 1.454\,s\text{^}2 + 2.22\,s + 1}, \; J = 5.80355*10\text{^}(-2) \quad (50)$$

3. AGTM matching for obtaining the numerator polynomial gives:

$$G_r(s) = \frac{0.07334\,s\text{^}2 + 0.1398\,s + 0.1}{0.2484s\text{^}3 + 1.454\,s\text{^}2 + 2.22\,s + 1}, \; J = 0.22047*10\text{^}(-2) \quad (51)$$

4. AGTM matching for obtaining the ROM gives:

$$G_r(s) = \frac{6.116\,s\text{^}2 + 29.99\,s + 55.07}{s\text{^}3 + 81.67\,s\text{^}2 + 754.7\,s + 550.7}, \; J = 0.00071*10\text{^}(-2) \quad (52)$$

In Figure 2, the step responses of the above transfer functions for example 1 are shown.

3.2 Example 2

The following high order plant transfer function is taken from [14] where,

$$G(s) = \frac{35\,s\text{^}7 + 1086\,s\text{^}6 + 13285\,s\text{^}5 + 82402\,s\text{^}4 + 278376\,s\text{^}3 + 511812\,s\text{^}2 + 482964\,s + 194480}{s\text{^}8 + 33\,s\text{^}7 + 437\,s\text{^}6 + 3017\,s\text{^}5 + 11870\,s\text{^}4 + 27470\,s\text{^}3 + 37492\,s\text{^}2 + 28880\,s + 9600}$$

The calculated reduced order transfer functions are:

1. Complete time moment matching gives:

$$G_r(s) = \frac{29\,s\text{^}3 + 53.31\,s\text{^}2 + 50.31s + 20.26}{1.236\,s\text{^}4 + 2.861\,s\text{^}3 + 3.905\,s\text{^}2 + 3.008\,s + 1} \quad J = 79.1010 \quad (53)$$

2. By matching three time moments and one Markov parameter:

$$G_r(s) = \frac{43.28\,s\text{^}3 + 53.31\,s\text{^}2 + 50.31\,s + 20.26}{1.236\,s\text{^}4 + 2.861\,s\text{^}3 + 3.905\,s\text{^}2 + 3.008\,s + 1} \quad J = 126.4811 \quad (54)$$

3. AGTM matching for numerator only gives:

$$G_r(s) = \frac{29\,s\text{^}3 + 53.31\,s\text{^}2 + 50.31\,s + 20.26)}{1.236s\text{^}4 + 2.861\,s\text{^}3 + 3.905\,s\text{^}2 + 3.008\,s + 1} \quad J = 40.9933 \quad (55)$$

4. AGTM matching for both the numerator and denominator gives:

$$G_r(s) = \frac{34.66\,s\text{^}3 + 189.5\,s\text{^}2 + 294.4\,s + 172.3}{s\text{^}4 + 7.232s\text{^}3 + 16.73\,s\text{^}2 + 19\,s + 8.506} \quad J = 0.0112 \quad (56)$$

In Figure 3, the step responses of the above transfer functions for Example 2, are shown.

From Figures 2 and 3, the ROMs obtained by different methods, one can conclude that the AGTM method for obtaining ROM gives the best result in matching the original transfer function response.

Conclusion

Various methods are described to reduce the order of higher-order continuous time, SISO or MIMO transfer. The methods are illustrated by solving several examples from

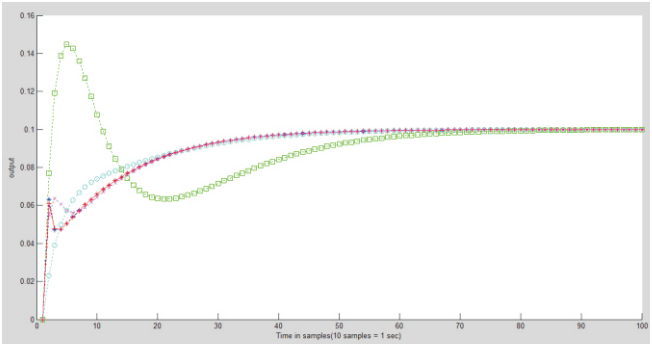

Figure 2. Step Response Comparison (Example 1)

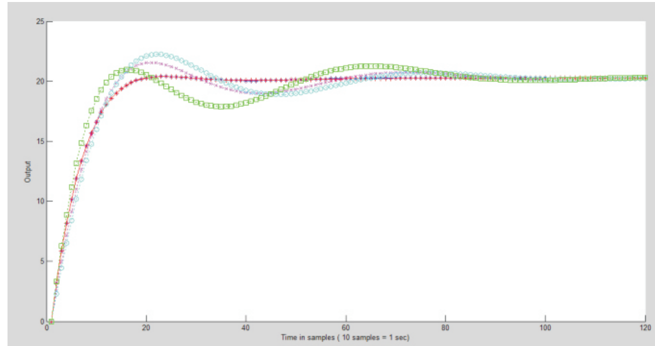

Figure 3. Step Response Comparisons (Example 2)

the literature. In the reduced order modeling method using "time moment and Markov parameter matching", the number of time moments and Markov parameters to be matched, and the optimal combination to be chosen is open to further investigation.

References

[1]. **Kim Y.C, Keel L.H, and Bhattacharyya S.P, (2003).** "Transient Response Control via Characteristic Ratio Assignment". *IEEE Transactions on Automatic Control*, Vol.48, pp.2238–2244.

[2]. **Kim Y.C, Keel L.H, and Bhattacharyya S.P., (2003).** "PID Controller Design with Time Response Specifications". *Proceedings of the American Control Conference*, pp. 5005–5010.

[3]. **Kim Y.C, Kim K.S, and Manabe S., (2006).** "Sensitivity of Time Response To Characteristic Ratios". *IEICE Transactions on Fundamentals of Electronics, Communications and Computer Sciences*, Vol.89, pp. 520–526.

[4]. **Kim Y, (2008).** "Transient Response Control". *17^{th} IFAC World Congress Tutorial Workshop on Advances in 3 Term Control*.

[5]. **Sarvesh, B, (1995).** "*Some New Methods For Reduced-order Modeling and Controller Design*". PhD thesis, Indian Institute of Technology, Kharagpur.

[6]. **Shamash, Y., (1974).** "Stable Reduced-order Models Using Paid- Type Approximations". *IEEE Transactions on Automatic Control*, Vol. 19, No. 5, pp. 615-616.

[7]. **Hickin, J. and Sinha, N. K., (1980).** "Model Reduction for Linear Multivariable Systems". *IEEE Transactions on Automatic Control*, Vol. 25, No. 6, pp. 1121-1127.

[8]. **Bandekas, D. V. and Papadopoulos, D. P., (1992).** "Time Moment and Paid Approximation Methods Applied to the Order Reducti9on of MIMO Linear Systems". *Journal of the Franklin Institute*, Vol. 329, No. 3, pp. 521-538.

[9]. **J. Van De Vegte, (1987).** "Classical Design, With Application To A 3x3 Turbofan Engine Model". *International Journal of Control*, Vol. 45, No. 1, pp. 1-16.

[10]. **Krishna Murthy V., and Seshadri V., (1978).** "Model reduction using the Routh stability criterion". *IEEE Transactions on Automatic Control*, Vol. AC-23, No.4.

[11]. **Ashoor N., and Singh V, (1982).** "A Note On Low-order Modeling". *IEEE Transactions on Automatic Control*, Vol.27, No.5.

[12]. **Pal J, Sarvesh B , and Ghosh M.K., (1995).** "A New Method For Model Order Reduction". *Journal of the IETE*, Vol. 41, Nos. 5 & 6, pp. 305-311.

[13]. **Chen C.T., (1987).** "Introduction To The Linear Algebraic Method For Control System Design". *IEEE Transactions on Control System Management*, Vol. 7, No. 5, pp. 36-42.

[14]. **C. T. Chen and B. Seo, (1989).** "Application of the Linear Algebraic Method For Control System Design". *IEEE Transactions on Control System Management*, Vol. 10, No. 1.

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