Common fixed point results in fuzzy metric spaces and their applications in dynamic programming

Abstract: In this paper we propose some common fixed point results for two pairs of self mappings satisfying CLR_g or CLR_{ST} properties and weak compatibility in fuzzy metric spaces. We also furnish some examples to support our results. The existence and uniqueness of solutions for certain system of functional equations arising in dynamic programming are also presented as an application.

Keywords: Fuzzy metric space, t - norm, t - conorm, weakly compatible mappings, (E. A) property, CLR_g property, CLR_{ST} property.

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1. Introduction:

Zadeh [21] first introduced the notion of fuzzy set in 1965. Fuzzy set assigns each object to a grade of membership between zero and one. Following the idea of fuzzy set, Kramosil and Michalek [15] extended the concept of metric space as fuzzy metric space. George and Veeramani [9] modified the concept of fuzzy metric space by imposing some stronger conditions using continuous t-norm and defined the hausdorff topology of fuzzy metric spaces. Gregori and Sapena[10] defined the concepts of convergent sequence, Cauchy sequence, completeness and compactness in fuzzy metric space.

Aamri and Moutawakil [1] defined the (E. A) property for self mappings which contains the class of non compatible as well as compatible mappings. Ali et al. [2] introduced common property (E. A). Mihet [16] introduced (E. A) property in fuzzy metric spaces. It is observed that (E. A) property and common property (E. A) require the closedness of the subspaces for the existence of fixed point. Sintunavarat and Kuman [20] defined the notion of common limit in the range property (CLR_g property). Imdad et al. [11] introduced the concept of CLR_{ST} property. It is to be noted that CLR_g and CLR_{ST} properties do not require closedness of range for the existence of common fixed point.

Jungck [12] first introduced commuting mappings in metric space. Generalizing the concept of commuting mapping Sessa [19] introduced the concept of weakly commuting mappings. Pant [17] defined weakly commuting maps in fuzzy metric space. Jungck [13] enlarged the class of noncommuting mappings by compatible mappings. In 1998, Jungck and Rhoades [14] introduced the notion of weakly compatible and showed that compatible maps are weakly compatible but the converse is not true.

Bellman [4] introduced the existence of solutions for some classes of functional equations arising in dynamic programming. Bellman and Lee [5] explained that the basic form of the functional equations in dynamic programming are as follows:

$$f(x) = \underbrace{opt}_{y \in D} \{H(x, y, f(T(x, y) \}, \ \forall x \in S$$

where opt denotes sup or inf, x and y denote the state and decision vectors, respectively, T stands for the transformation of the process, and $f(x)$ represents the optimal return function with the initial state x. Thereafter many authors ([3], [6], [7], [8]) have done remarkable work on the existence and uniqueness of solution and common solution for functional equations in dynamic programming.

In this paper, we prove some common fixed point theorems for two pairs of weakly compatible mappings in fuzzy metric space using CLR_{ST} and CLR_g properties. We also furnish some examples to support our results. The existence and uniqueness of solutions for certain system of functional equations arising in dynamic programming are also presented as an application.

2. Preliminaries:

Schweizer and Sklar [18] defined the continuous t-norm as,

Definition 2.1 [18]: A binary operation $*: [0,1] \times [0,1] \rightarrow [0,1]$ is called continuous t-norm if it satisfies the following conditions:

1) $*$ is commutative and associative;
2) $*$ is continuous;
3) $a * 1 = a$, for all $a \in [0, 1]$;

4) $a * b \le c * d$, whenever $a \le c$ and $b \le d$, for all a, b, c, d $\in$ [0, 1].

George and Veeramani[9] extended the concept of fuzzy metric space defined by Kramosil and Michalek[11] with the help of continuous t-norm to introduce housdorff topology in a fuzzy metric space.

Definition 2.2[9]: The 3 - tuple $(X, M, *)$ is said to be a fuzzy metric space (FMS) if, X is a non empty set, $*$ is a continuous t - norm, M is a fuzzy set on $X \times X \times (0, \infty)$ satisfying the following conditions:

1) $M(x, y, t) > 0$;
2) $M(x, y, t) = 1$ if and only if $x = y$;
3) $M(x, y, t) = M(y, x, t)$;
4) $M(x, y, t) * M(y, z, s) \le M(x, z, t + s)$;
5) $M(x, y, \cdot): (0, \infty) \to [0, 1]$ is continuous, for all $x, y, z \in X$ and $s, t > 0$.

M(x, y, t) is considered as the degree of nearness of x and y with respect to t.

Example 2.3 [9]:

Let (X, d) be a metric space, t- norm is defined as $a * b = \min\{a, b\}$ $\forall x, y \in X$ and $t > 0$.

$$M_d(x, y, t) = \frac{t}{t + d(x, y)},$$

then (X, M, $*$) is a Fuzzy metric space.

Pant [17] introduced the concept of weakly commuting maps in fuzzy metric space as,

Definition 2.4 [17]: Two self mappings f and g of a fuzzy metric space $(X, M, *)$ are said to be weakly commuting if,

$M(fgx, gfx, t) \ge M(fx, gx\ t)$, for all $x \in X$, $t > 0$.

Jungck [13] extended the class of noncommuting mappings by defining compatible mappings.

Definition 2.5 [13]: Two self mappings f and g of a fuzzy metric space (X, M, ∗) are said to be compatible if,

$$\lim_{n\to\infty} M(fgx_n, gfx_n, t) = 1,$$

whenever $\{x_n\}$ is a sequence in X such that,

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = x,$$

for some $x \in X$, $t > 0$.

Jungck and Rhoades [14] defined weakly compatible and showed that compatible maps are weakly compatible but the converse is not true.

Definition 2.6[14]: Two self mappings f and g of a fuzzy metric space (X, M, ∗) are said to be weakly compatible if they commute at their coincidence point that is, if fx = gx for some $x \in X$, then

M (fgx, gfx, t) =1

It is to be noted that if two mappings are compatible then they are weakly compatible, but converse is not true.

Mihet first defined the (E. A) property in fuzzy metric space,

Definition 2.7 [16] : Two self mappings f and g of a fuzzy metric space (X, M, ∗) are said to satisfy the (E. A) property if there exist a sequence $\{x_n\}$ in X such that for all $t > 0$,

$$\lim_{n\to\infty} M(fx_n, gx_n, t) = 1.$$

To relax the closedness condition, Sintunavarat and Kuman [20] defined the notion of common limit in the range property (CLR_g property).

Definition 2.8 [20] : Two self mappings f and g of a fuzzy metric space (X, M, ∗) are said to satisfy the common limit in the range of g (CLRg) property if there exist a sequence $\{x_n\}$in X such that,

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} g\,x_n = gu, \text{ for some } u \in X.$$

Definition 2.9[2]: Two pairs (A, S) and (B, T) of self mappings of a fuzzy metric space (X, M, ∗) are said to be share the (E. A) property if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that for all t > 0,

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = z, \text{ for some z in X.}$$

Imdad et al. [11] introduced the concept of CLR_{ST} property.

Definition 2.10[11]**:** Two pairs (A, S) and (B, T) of self mappings of a fuzzy metric space (X, M, ∗) are said to satisfy the (CLR_{ST}) property if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that for all t > 0,

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = z,$$

Where, $z \in S(X) \cap T(X)$.

Main results:

Definition 3.1:

Let Ψ be the collection of all functions α: [0, 1] → [0, ∞) where α is strictly decreasing function s. t., α (1) = 0 and α (0) < ∞.

Definition 3.2:

Let Φ be the collection of all functions ϕ: [0, ∞) → [0,∞) which are right continuous and $\phi(t) < t$, for all t > 0.

Lemma 3.3:

Let ϕ: [0,∞) →[0,∞) be a function such that $\phi \in \Phi$ then for all t ≥ 0,

If $t \leq \phi(t)$, then t = 0.

Theorem 3.4:

Let A, B, S and T be four self mappings of a fuzzy metric space (X, M, ∗) such that,

I. The pairs (A, S) and (B, T) share the (CLR_{ST}) property.

II. Both the pairs (A, S) and (B, T) are weakly compatible.

III. $\alpha(M(Ax, By, t)) \leq$

$$\phi\Big(\max \Big\{\alpha(M(Sx,Ty,t)), \alpha(M(Sx,Ax,t)), \alpha(M(By,Ty,t)), \tfrac{1}{2}\Big(\alpha\big(M(Ax,Sx,t)\big) + \alpha\big(M(By,Ty,t)\big)\Big), \tfrac{1}{2}\Big(\alpha\big(M(Ax,Sx,t)\big) + \alpha\big(M(Sx,Ty,t)\big)\Big), \tfrac{1}{2}\Big(\alpha\big(M(By,Ty,t)\big) + \alpha\big(M(Sx,Ty,t)\big)\Big), \tfrac{1}{2}\Big(\alpha\big(M(Sx,By,t)\big) + \alpha\big(M(Ty,Ax,t)\big)\Big\}\Big)$$

for all x, y ∈ X and t > 0, then A, B, S and T have a unique common fixed point in X.

Proof:

Since the pairs (A, S) and (B, T) share the CLR_{ST} property, there exist two sequences $\{x_n\}$ and $\{y_n\}$ such that,

$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ty_n = \lim_{n\to\infty} By_n = z$, where z ∈ S(X) ∩ T(X).

As z ∈ S(X), there exists a point x ∈ X such that Sx = z.

We first show that Ax = Sx.

From the inequality III,

$$\alpha(M(Ax, By_n,\ t)) \leq$$

$$\phi\ (\ \max\{\alpha(M(Sx, Ty_n, t)), \alpha(M(Sx, Ax, t)), \alpha(M(By_n, Ty_n, t)), \tfrac{1}{2}\left(\alpha\left(M(Ax, Sx, t)\right) + \alpha\left(M(By_n, Ty_n, t)\right)\right), \tfrac{1}{2}(\alpha\left(M(Ax, Sx, t)\right) + \alpha\left(M(Sx, Ty_n, t)\right), \tfrac{1}{2}\left(\alpha\left(M(By_n, Ty_n, t)\right) + \alpha\left(M(Sx, Ty_n, t)\right)\right), \tfrac{1}{2}(\alpha\left(M(Sx, By_n, t)\right) + \alpha\left(M(Ty_n, Ax, t)\right)\})$$

Now taking $\lim_{n\to\infty}$

$$\alpha(M(Ax, Bz,\ t)) \leq$$

$$\phi\ (\ \max\{\alpha(M(Sx, Tz, t)), \alpha(M(Sx, Ax, t)), \alpha(M(Bz, Tz, t)), \tfrac{1}{2}\left(\alpha\left(M(Ax, Sx, t)\right) + \alpha\left(M(Bz, Tz, t)\right)\right), \tfrac{1}{2}(\alpha\left(M(Ax, Sx, t)\right) + \alpha\left(M(Sx, Tz, t)\right), \tfrac{1}{2}\left(\alpha\left(M(Bz, Tz, t)\right) + \alpha\left(M(Sx, Tz, t)\right)\right), \tfrac{1}{2}(\alpha\left(M(Sx, Bz, t)\right) + \alpha\left(M(Tz, Ax, t)\right)\})$$

$$\alpha(M(Ax, z,\ t)) \leq\ \phi\ (\ \max\{\alpha(M(z, z, t)), \alpha(M(z, Ax, t)), \alpha(M(z, z, t)), \tfrac{1}{2}\left(\alpha\left(M(Ax, z, t)\right) + \alpha\left(M(z, z, t)\right)\right), \tfrac{1}{2}(\alpha\left(M(z, z, t)\right) + \alpha\left(M(z, z, t)\right), \tfrac{1}{2}\left(\alpha\left(M(z, z, t)\right) + \alpha\left(M(z, z, t)\right)\right), \tfrac{1}{2}(\alpha\left(M(z, z, t)\right) + \alpha\left(M(z, z, t)\right)\})$$

$$\alpha(M(Ax, z,\ t)) \leq\ \phi(\max\{\alpha(1), \alpha\left(M(z, Ax, t)\right), \alpha(1), \tfrac{1}{2}\left(\alpha\left(M(Ax, z, t)\right) + \alpha(1)\right), \tfrac{1}{2}\left(\alpha(1) + \alpha(1)\right), \tfrac{1}{2}\left(\alpha(1) + \alpha(1)\right), \tfrac{1}{2}\left(\alpha(1) + \alpha(1)\right)\})$$

$$\alpha(M(Ax, z,\ t)) \leq\ \phi\ (\ \max\{\alpha(1), \alpha(M(z, Ax, t)), \alpha(1), \tfrac{1}{2}\left(\alpha\left(M(Ax, z, t)\right) + \alpha(1)\right), \tfrac{1}{2}(\alpha(1) + \alpha(1)), \tfrac{1}{2}\left(\alpha(1) + \alpha(1)\right), \tfrac{1}{2}(\alpha(1) + \alpha(1)\})$$

$$\alpha(M(Ax, z,\ t)) \leq\ \phi\ (\ \max\{0, \alpha\left(M(z, Ax, t)\right), 0, \tfrac{1}{2}\left(\alpha\left(M(Ax, z, t)\right) + 0\right), \tfrac{1}{2}(0 + 0), \tfrac{1}{2}(0 + 0), \tfrac{1}{2}(0 + 0)\})$$

$$\alpha(M(Ax, z,\ t)) \leq\ \phi\ (\alpha(M(z, Ax, t))$$

hence, using Lemma 3.3 $Ax = z$.

therefore, Ax = Sx = z. which shows that x is a coincidence point of the pair (A, S).

As z ∈ T(X) there exist a point y ∈ T(X) such that, Ty = z.

Next we show that By = Ty.

From inequality III,

$$\alpha(M(Ax,By,\ t)) \leq$$

$$\phi\ (\ \max\{\alpha(M(Sx,Ty,t)),\alpha(M(Sx,Ax,t)),\alpha(M(By,Ty,t)),\frac{1}{2}\Big(\alpha\big(M(Ax,Sx,t)\big)+\alpha\big(M(By,Ty,t)\big)\Big),$$

$$\frac{1}{2}(\alpha\big(M(Ax,Sx,t)\big)+\alpha\big(M(Sx,Ty,t)\big),\frac{1}{2}\Big(\alpha\big(M(By,Ty,t)\big)+\alpha\big(M(Sx,Ty,t)\big)\Big),\frac{1}{2}(\alpha\big(M(Sx,By,t)\big)+\alpha\big(M(Ty,Ax,t)\big)\})$$

Hence,

$$\alpha(M(z,By,t))$$

$$\leq\ \phi\ (\ \max\{\alpha(M(z,z,t)),\alpha(M(z,z,t)),\alpha(M(By,z,t)),\frac{1}{2}\Big(\alpha\big(M(z,z,t)\big)+\alpha\big(M(By,z,t)\big)\Big),\frac{1}{2}(\alpha\big(M(z,z,t)\big)$$

$$+\alpha\big(M(z,z,t)\big),\frac{1}{2}\Big(\alpha\big(M(By,z,t)\big)+\alpha\big(M(z,z,t)\big)\Big),\frac{1}{2}(\alpha\big(M(z,By,t)\big)+\alpha\big(M(z,z,t)\big)\}$$

$$\leq\ \phi\ (\ \max\{\alpha(1),\alpha(1),\alpha(M(By,z,t)),\frac{1}{2}\Big(\alpha(1)+\alpha\big(M(By,z,t)\big)\Big),$$

$$\frac{1}{2}(\alpha(1)\quad+\alpha(1),\frac{1}{2}\Big(\alpha\big(M(By,z,t)\big)+\alpha(1)\Big),\frac{1}{2}(\alpha\big(M(z,By,t)\big)+\alpha(1)\})$$

$$\leq\ \phi\ (\ \max\{0,0,\alpha\big(M(By,z,t)\big),\frac{1}{2}\Big(0+\alpha\big(M(By,z,t)\big)\Big),\frac{1}{2}(0+0),\frac{1}{2}\big(\alpha\big(M(By,z,t)\big)+0\big),\frac{1}{2}\big(\alpha\big(M(z,By,t)\big)+0\big)\})$$

Hence,

$\alpha(M(z, By, t)) \leq \phi\ (\ \alpha\big(M(By, z, t)\big)$

hence, using Lemma 3.3

By = Ty = z. which shows that y is a coincidence point of the pair (B, T).

Let the pair A and S be weakly compatible then, Ax = Sx.

Hence, Az = ASx = SAx = Sz.

Now we show that z is a common fixed point of A and S,

From inequality III,

$$\alpha(M(Ax, By,\ t)) \leq \phi\ (\ \max\{\alpha(M(Sx, Ty, t)), \alpha(M(Sx, Ax, t)), \alpha(M(By, Ty, t)), \tfrac{1}{2}\Big(\alpha\big(M(Ax, Sx, t)\big) + \alpha\big(M(By, Ty, t)\big)\Big),$$
$$\frac{1}{2}(\alpha\big(M(Ax, Sx, t)\big) + \alpha\big(M(Sx, Ty, t)\big), \frac{1}{2}\Big(\alpha\big(M(By, Ty, t)\big) + \alpha\big(M(Sx, Ty, t)\big)\Big), \frac{1}{2}(\alpha\big(M(Sx, By, t)\big)$$
$$+ \alpha\big(M(Ty, Ax, t)\big)\})$$

$$\alpha(M(z, By, t)) \leq \phi\ (\ \max\{\alpha(M(z, z, t)), \alpha(M(z, z, t)), \alpha(M(By, z, t)), \tfrac{1}{2}\Big(\alpha\big(M(z, z, t)\big) + \alpha\big(M(By, z, t)\big)\Big), \tfrac{1}{2}(\alpha\big(M(z, z, t)\big) +$$
$$\alpha\big(M(z, z, t)\big), \tfrac{1}{2}\Big(\alpha\big(M(By, z, t)\big) + \alpha\big(M(z, z, t)\big)\Big), \tfrac{1}{2}(\alpha\big(M(z, By, t)\big) + \alpha\big(M(z, z, t)\big)\}),$$ hence

$\alpha(M(z, By, t))$

$$\leq \phi\ (\ \max\{\alpha(1), \alpha(1), \alpha(M(By, z, t)), \frac{1}{2}\Big(\alpha(1) + \alpha\big(M(By, z, t)\big)\Big), \frac{1}{2}(\alpha(1) + \alpha(1), \frac{1}{2}\Big(\alpha\big(M(By, z, t)\big) + \alpha(1)\Big), \frac{1}{2}\Big(\alpha\big(M(z, By, t)\big) + \alpha(1)\Big)\})$$
$$\leq \phi\ (\ \max\{0, 0, \alpha\big(M(By, z, t)\big), \frac{1}{2}\Big(0 + \alpha\big(M(By, z, t)\big)\Big), \frac{1}{2}(0 + 0), \frac{1}{2}(\alpha\big(M(By, z, t)\big) + 0), \frac{1}{2}(\alpha\big(M(z, By, t)\big) + 0)\})$$

Hence, $\alpha(M(z,By,t)) \leq \phi\,(\alpha(M(By,z,t))$

Now using Lemma 3.3,

By = Ty = z. therefore z is a common fixed point of the pair (B, T).

Hence, z is a common fixed point of the pairs (A, S) and (B, T).

Now we show uniqueness of the fixed point,

Let z and w be two common fixed points of the pairs (A, S) and (B, T).

From inequality III,

$$\alpha(M(Az,Bw,t)) \leq \phi\,(\max\{\alpha(M(Sz,Tw,t)),\alpha(M(Sz,Az,t)),\alpha(M(Bw,Tw,t)),\tfrac{1}{2}\big(\alpha(M(Az,Sz,t))+\alpha(M(Bw,Tw,t))\big),$$

$$\frac{1}{2}(\alpha(M(Az,Sz,t))+\alpha(M(Sz,Tw,t)),\frac{1}{2}\big(\alpha(M(Bw,Tw,t))+\alpha(M(Sz,Tw,t))\big),\frac{1}{2}(\alpha(M(Sz,Bw,t))$$

$$+\alpha(M(Tw,Az,t))\})$$

$$\alpha(M(z,w,t))$$

$$\leq \phi\,(\max\{\alpha(M(z,w,t)),\alpha(M(z,z,t)),\alpha(M(w,w,t)),\frac{1}{2}\big(\alpha(M(z,z,t))+\alpha(M(w,w,t))\big),\frac{1}{2}(\alpha(M(z,z,t))$$

$$+\alpha(M(z,z,t)),\frac{1}{2}\big(\alpha(M(w,w,t))+\alpha(M(z,w,t))\big),\frac{1}{2}(\alpha(M(z,w,t))+\alpha(M(z,z,t))\})$$

$$\leq \phi\,(\max\{\alpha(M(z,w,t)),\alpha(1),\alpha(1),\frac{1}{2}(\alpha(1)+\alpha(1)),\frac{1}{2}(\alpha(1)+\alpha(1),\frac{1}{2}(\alpha(1)+\alpha(M(z,w,t))),\frac{1}{2}\big(\alpha(M(z,w,t))+\alpha(1)\big)\})$$

$$\leq \phi\,(\max\{\alpha(M(z,w,t)),0,0,\tfrac{1}{2}(0+0),\tfrac{1}{2}(0+0),\tfrac{1}{2}(0+\alpha(M(z,w,t))),\tfrac{1}{2}\big(\alpha(M(z,w,t))+0\big)\})$$

Hence, $\alpha(M(z,w,t)) \leq \phi\,(\alpha(M(z,w,t))$

Using Lemma 3.2, z = w.

Therefore z is a unique common fixed point of the pairs (A, S) and (B, T).

Theorem 3.5:

Let A, B, S and T be four self mappings of a fuzzy metric space (X, M, $*$) such that,

I. If the pairs (A, S) satisfies the (CLR_S) property or (B, T) satisfies the (CLR_T) property.

II. $A(X) \subset S(X)$ or $B(X) \subset T(X)$.

III. T(X) or S(X) is a closed subset of X.

IV. Both the pairs (A, S) and (B, T) are weakly compatible.

V. Ax_n converges for every sequence x_n in X whenever Sx_n converges or By_n converges for every sequence y_n in X whenever Ty_n converges.

VI. $\alpha(M(Ax, By, t)) \leq$

$$\phi\left(\max\left(\alpha\big(M(Sx,Ty,t)\big), \alpha\big(M(Sx,Ax,t)\big), \alpha\big(M(By,Ty,t)\big), \frac{1}{2}\Big(\alpha\big(M(Ax,Sx,t)\big) + \alpha\big(M(By,Ty,t)\big)\Big), \frac{1}{2}\Big(\alpha\big(M(Ax,Sx,t)\big) + \alpha\big(M(Sx,Ty,t)\big)\Big), \frac{1}{2}\Big(\alpha\big(M(By,Ty,t)\big) + \alpha\big(M(Sx,Ty,t)\big)\Big), \frac{1}{2}\Big(\alpha\big(M(Sx,By,t)\big) + \alpha\big(M(Ty,Ax,t)\big)\Big)\right)\right).$$

for all x, y $\in$ X and t > 0, then A, B, S and T have a unique common fixed point in X.

Proof:

Since the pair (A, S) satisfies the CLR_s property, there exists a sequence x_n in X such that,

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = x, for\ some\ x \in X.$$

Since A(X) is a subset of T(X),

$Ax_n = Ty_n$.

Since T(X) is a closed subspace of X,

$$\lim_{n\to\infty} Ty_n = \lim_{n\to\infty} Ax_n = x$$

Where x ∈ S(X) ∩ T(X).

Therefore, $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ty_n$

Now we show that, $\lim_{n\to\infty} By_n = z$

Let $\lim_{n\to\infty} By_n = l$

From inequality III,

$$\alpha(M(Ax_n, By_n, t)) \leq$$
$$\phi\ (\max\{\alpha(M(Sx_n, Ty_n, t)), \alpha(M(Sx_n, Ax_n, t)), \alpha(M(By_n, Ty_n, t)), \tfrac{1}{2}\Big(\alpha\big(M(Ax_n, Sx_n, t)\big) +$$
$$\alpha\big(M(By_n, Ty_n, t)\big)\Big), \tfrac{1}{2}(\alpha\big(M(Ax_n, Sx_n, t)\big) + \alpha\big(M(Sx_n, Ty_n, t)\big), \tfrac{1}{2}\Big(\alpha\big(M(By_n, Ty_n, t)\big) + \alpha\big(M(Sx_n, Ty_n, t)\big)\Big), \tfrac{1}{2}(\alpha\big(M(Sx_n, By_n, t)\big) +$$
$$\alpha\big(M(Ty_n, Ax_n, t)\big)\})$$

taking the limit n→∞ we have,

$$\alpha(M(z, l, t)) \leq$$
$$\phi\ (\max\{\alpha\big(M(z, z, t)\big), \alpha\big(M(z, z, t)\big), \alpha\big(M(l, z, t)\big), \tfrac{1}{2}\Big(\alpha\big(M(z, z, t)\big) + \alpha\big(M(l, z, t)\big)\Big), \tfrac{1}{2}\Big(\alpha\big(M(z, z, t)\big) + \alpha\big(M(z, z, t)\big), \tfrac{1}{2}\Big(\alpha\big(M(l, z, t)\big) +$$
$$\alpha\big(M(z, z, t)\big)\Big), \tfrac{1}{2}\Big(\alpha\big(M(z, l, t)\big) + \alpha\big(M(z, z, t)\big)\}\Big), \text{hence}$$

$$\alpha(M(z, l, t)) \leq$$
$$\phi\ (\max\{\alpha(M(z, z, t)), \alpha(M(z, z, t)), \alpha(M(l, z, t)), \tfrac{1}{2}\Big(\alpha\big(M(z, z, t)\big) + \alpha\big(M(l, z, t)\big)\Big), \tfrac{1}{2}(\alpha\big(M(z, z, t)\big) + \alpha\big(M(z, z, t)\big), \tfrac{1}{2}\Big(\alpha\big(M(l, z, t)\big) +$$
$$\alpha\big(M(z, z, t)\big)\Big), \tfrac{1}{2}(\alpha\big(M(z, l, t)\big) + \alpha\big(M(z, z, t)\big)\})$$

$$\alpha(M(z,l,\ t)) \leq\ \phi\ (\ \max\{\alpha(1),\alpha(1),\alpha(M(l,z,t)),\tfrac{1}{2}\Big(\alpha(1)+\alpha\big(M(l,z,t)\big)\Big),\tfrac{1}{2}(\alpha(1)+\alpha(1),\ \tfrac{1}{2}\Big(\alpha\big(M(l,z,t)\big)+\alpha(1)\Big),\tfrac{1}{2}\Big(\alpha\big(M(z,l,t)\big)+\alpha(1)\Big)\})$$

$$\leq\ \phi\ (\ \max\{0,0,\alpha\big(M(l,z,t)\big),\tfrac{1}{2}(0+0),\tfrac{1}{2}(0),\tfrac{1}{2}\big(\alpha\big(M(l,z,t)\big)+0\big),\tfrac{1}{2}\big(\alpha\big(M(z,l,t)\big)+0\big)\})$$

$$\leq\ \phi\ (\alpha(M(l,z,t))$$

hence, using $z = l$.

hence the pair (A, S) and (B, T) shares CLR_{ST} property.

hence using theorem 3.3 (A, S) and (B, T) have common fixed points.

Corollary 3.6:

Let A, B, S and T be four self mappings of a fuzzy metric space (X, M, $*$) such that,

I. $\alpha(M(Ax,By,t)) \leq$

$$\phi\ (\ \max\ (\alpha\big(M(Sx,Ty,t)\big),\alpha\big(M(Sx,Ax,t)\big),\alpha\big(M(By,Ty,t)\big),\tfrac{1}{2}\Big(\alpha\big(M(Ax,Sx,t)\big)+\alpha\big(M(By,Ty,t)\big)\Big),\tfrac{1}{2}\Big(\alpha\big(M(Ax,Sx,t)\big)+\alpha\big(M(Sx,Ty,t)\big),\tfrac{1}{2}\Big(\alpha\big(M(By,Ty,t)\big)+\alpha\big(M(Sx,Ty,t)\big)\Big),\tfrac{1}{2}\Big(\alpha\big(M(Sx,By,t)\big)+\alpha\big(M(Ty,Ax,t)\big)\Big)\Big).$$

II. The pairs (A, S) and (B, T) satisfy the common property (E.A);

III. S(X) and T (X) are closed subsets of X.

IV. both pairs (A, S) and (B, T) are weakly compatible.

Then (A, S) and (B, T) have a unique common fixed point.

Proof: If the pairs (A, S) and (B, T) satisfy the common property (E.A), and at the same time, S(X) and T (X) are closed subsets of X, then the pairs (A, S) and (B, T) share the (CLR_{ST}) property. Hence using theorem 3.3 the above corollary can be proved.

Example 3.7:

Let (X, M, $*$) be a fuzzy metric space.

Let X = [2, 11] where M(x, y, t) = $\frac{t}{t+|x-y|}$ and self mappings A, B, S, T are defined in X as,

$$Ax = \begin{cases} 2, & if\ x \in \{2\} \cup (5, 11), \\ 5, & if\ x \in (2,\ 5]: \end{cases}$$

$$Bx = \begin{cases} 2, & if\ x \in \{2\} \cup (5, 11), \\ 6, & if\ x \in (2,\ 5]: \end{cases}$$

$$Sx = \begin{cases} 2, & if\ x = 2, \\ 5, & if\ x \in (2,\ 5] \\ \frac{x-1}{2}, & if\ x \in (5, 11) \end{cases}$$

$$Tx = \begin{cases} 2, & if\ x = 2, \\ 6, & if\ x \in (2,\ 5] \\ x-3, & if\ x \in (5, 11) \end{cases}$$

It is clear that Ax is closed subset of Sx and Bx is closed subset of Tx. Let $\{x_n\}$= {5+$\frac{1}{n}$ } and $\{y_n\}$= 2 and

$\phi(t) = \kappa t\ where\ 0 < \kappa < 1\ for\ all\ t \geq 0$, since all the conditions of theorem 3.4 are satisfied hence 2 is the common fixed point of A, B, S and T.

4. Application in dynamic programming:

In this section, we prove the existence and uniqueness of solutions for a certain system of functional equations arising in dynamic programming. We consider the following system of functional equations, to which a multistage process can be reduced

$$q(x) = sup_{y\in D}\{f(x, y) + G_i(x, y, q(\tau(x, y)))\},\ x \in W,\ i \in \{1, 2, 3, 4\},$$

where U and V are Banach spaces, W ⊆ U is a state space, D ⊆ V is a decision space, while τ : W × D → W, f : W × D → R, G_i : W × D × R → R are mappings, i ∈ {1, 2, 3, 4}.

Let X be the set of all bounded real valued functions on W, for h ∈ X, let $\|h\| = sup_{x\in W}|h(x)|$.

Clearly (X, ‖. ‖) is a Banach space, and the convergence in this space is uniform. Therefore, if {h_n} is a Cauchy sequence in X, then it converges uniformly to a function $h^* \in X$. The respective distance function is denoted by d.

Further, consider operators A, B, S, T: X → X given by,

$$\begin{cases} Ah(x) = sup_{y\in D}\{f(x,y) + G_1(x,y,h(\tau(x,y)))\} \\ Bh(x) = sup_{y\in D}\{f(x,y) + G_2(x,y,h(\tau(x,y)))\} \\ Sh(x) = sup_{y\in D}\{f(x,y) + G_3(x,y,h(\tau(x,y)))\} \\ Th(x) = sup_{y\in D}\{f(x,y) + G_4(x,y,h(\tau(x,y)))\} \end{cases} \quad \ldots\ldots\ldots\ldots\ldots\{4.1\}$$

For h ∈ X and x ∈ W.

Theorem 4.1:

Let A, B, S, T be self mappings in X, satisfying the equation (4.1), suppose they satisfy the following properties:

I) The functions $G_i: W \times D \times R \to R\ where\ i \in \{1,2,3,4\}$ satisfy

$$e^{-\frac{t}{sup_{x\in W}sup_{y\in D}|G_1(x,y,h(x)) - G_1(x,y,h(x))|}}$$

$$\leq \phi(\max\left\{\alpha\big(M(Sh,Tk,t)\big), \alpha\big(M(Sh,Ah,t)\big), \alpha\big(M(Tk,Bk,t)\big), \tfrac{1}{2}\Big(\alpha\big(M(Ah,Sh,t)\big) + \alpha\big(M(Bk,Tk,t)\big)\Big), \tfrac{1}{2}\Big(\alpha\big(M(Ah,Sh,t)\big) + \alpha\big(M(Sh,Tk,t)\big)\Big), \tfrac{1}{2}\Big(\alpha\big(M(Tk,Sh,t)\big) + \alpha\big(M(Bk,Tk,t)\big)\Big), \tfrac{1}{2}\Big(\alpha\big(M(Sh,Bk,t)\big) + \alpha\big(M(Tk,Ah,t)\big)\Big)\right\})$$

for all h, k ∈ X and t ∈ [0, 1] and α(t) = 1− t.

II) f: W × D → R and G_i: W × D × R → R are bounded functions, for $i \in \{1,2,3,4\}$.

III) there exist sequences {h_n} and {k_n} in X and h^* ∈ X such that,

$$\lim_{n\to\infty} Ah_n = \lim_{n\to\infty} Sh_n = \lim_{n\to\infty} Bk_n = \lim_{n\to\infty} Tk_n = h^*.$$

IV) ASh = SAh, whenever Ah = Sh for some h ∈ X;

V) BTk = TBk, whenever Bk = Tk for some k ∈ X.

Then the system of functional equations (5.1) has a unique bounded solution.

Proof:

Let $M(h, k, t) = \begin{cases} 1 - \exp\left\{-\frac{t}{d(h,k)}\right\} if\ 0 < t \leq d(h,k), h \neq k \\ 1 \quad otherwise \end{cases}$

Where h, k ∈ X and $a * b = \min\{a, b\}$ for a, b ∈ [0, 1], then (X, M, $*$) is a complete fuzzy metric space.

Since the pairs (A, S) and (B, T) share CLR_{ST} property, now let x ∈ W, h, k ∈ X and $\varepsilon > 0$ such that,

$$Ah(x) < f(x, y_1) + G_1(x, y_1, h(\tau(x, y_1))) + \varepsilon \quad \text{....(1)}$$

$$Ah(x) \geq f(x, y_2) + G_1(x, y_2, h(\tau(x, y_2))) \quad \text{....(2)}$$

$$Bk(x) < f(x, y_2) + G_2(x, y_2, k(\tau(x, y_2))) + \varepsilon \quad \text{....(3)}$$

$$Bk(x) \geq f(x, y_1) + G_2(x, y_1, k(\tau(x, y_1))) \quad \text{....(4)}$$

Using equation (1) and (2) we have,

$$Ah(X) - Bk(X) < G_1(x, y_1, h(\tau(x, y_1))) + \varepsilon - G_2(x, y1, k(\tau(x, y_1))$$

$$\leq |G_1(x, y_1, h(\tau(x, y_1))) - G2(x, y1, k(\tau(x, y_1)| + \varepsilon$$

$$\leq sup_{y \in D}|G_1(x, y, h(\tau(x, y))) - G_2(x, y, k(\tau(x, y)| + \varepsilon \quad \text{....(5)}$$

Similarily using (2) and (4) we have

$$Bk(X) - Ah(X) < sup_{y \in D}|G_1(x, y, k(\tau(x, y))) - G_2(x, y, h(\tau(x, y)| \quad \text{....(6)}$$

From (5) and (6) we can get,

$Ah(X) - Bk(X) < sup_{y \in D}|G_1(x, y, h(\tau(x, y))) - G_2(x, y, k(\tau(x, y)|$, therefore

$$d(Ah, Bk) \leq sup_{x \in W} sup_{y \in D} |G_1(x, y, h(\tau(x, y))) - G_2(x, y, k(\tau(x, y)| \quad \text{....(7)}$$

using equation (7) and property (I) it can be easily proved that,

$$\alpha\big(M(Ah,Bk,t)\big)$$

$$\leq \phi(\max\Big\{\alpha\big(M(Sh,Tk,t)\big),\alpha\big(M(Sh,Ah,t)\big),\alpha\big(M(Tk,Bk,t)\big),\frac{1}{2}\Big(\alpha\big(M(Ah,Sh,t)\big)+\alpha\big(M(Bk,Tk,t)\big)\Big),\frac{1}{2}\Big(\alpha\big(M(Ah,Sh,t)\big)$$

$$+\alpha\big(M(Sh,Tk,t)\big)\Big),\frac{1}{2}\Big(\alpha\big(M(Tk,Sh,t)\big)+\alpha\big(M(Bk,Tk,t)\big)\Big),\frac{1}{2}\Big(\alpha\big(M(Sh,Bk,t)\big)+\alpha\big(M(Tk,Ah,t)\big)\Big)\Big\})$$

Since the pair (A, S) and (B, T) are weakly compatible, hence using theorem 2.3, A, B, S and T have a common fixed point, hence the system of functional equation has a unique bounded solution.

5. Conclusion:

In this paper, we have proved some common fixed point theorems for two pairs of weakly compatible mappings in fuzzy metric space using the CLR_g and CLR_{ST} properties. Since common limit in the range properties have been used therefore closedness condition has been relaxed. Continuity condition is also not required. Some examples are given to support the results. The existence and uniqueness of solutions for certain system of functional equations arising in dynamic programming are also presented as an application

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